Compressing Large Data Sets with Geometry

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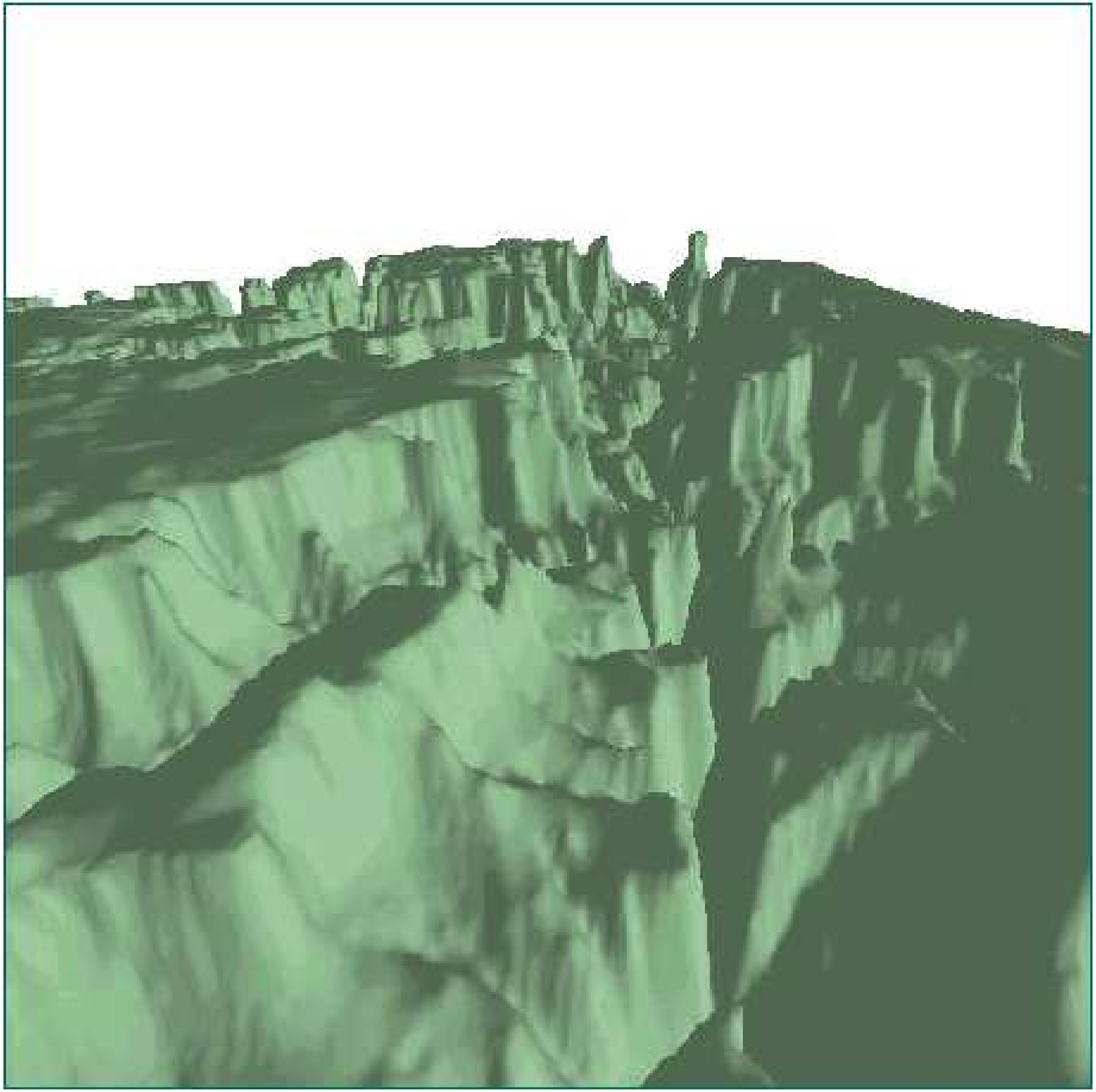
Large Data Sets with Geometry

- Digital Elevation Maps
- Medical Imagery
- Computer Aided Design
- Reverse Engineering
- Steering Large Scale Computation

Surface: Graph of $2d$- function



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- if resolution is unacceptable client asks for more bits

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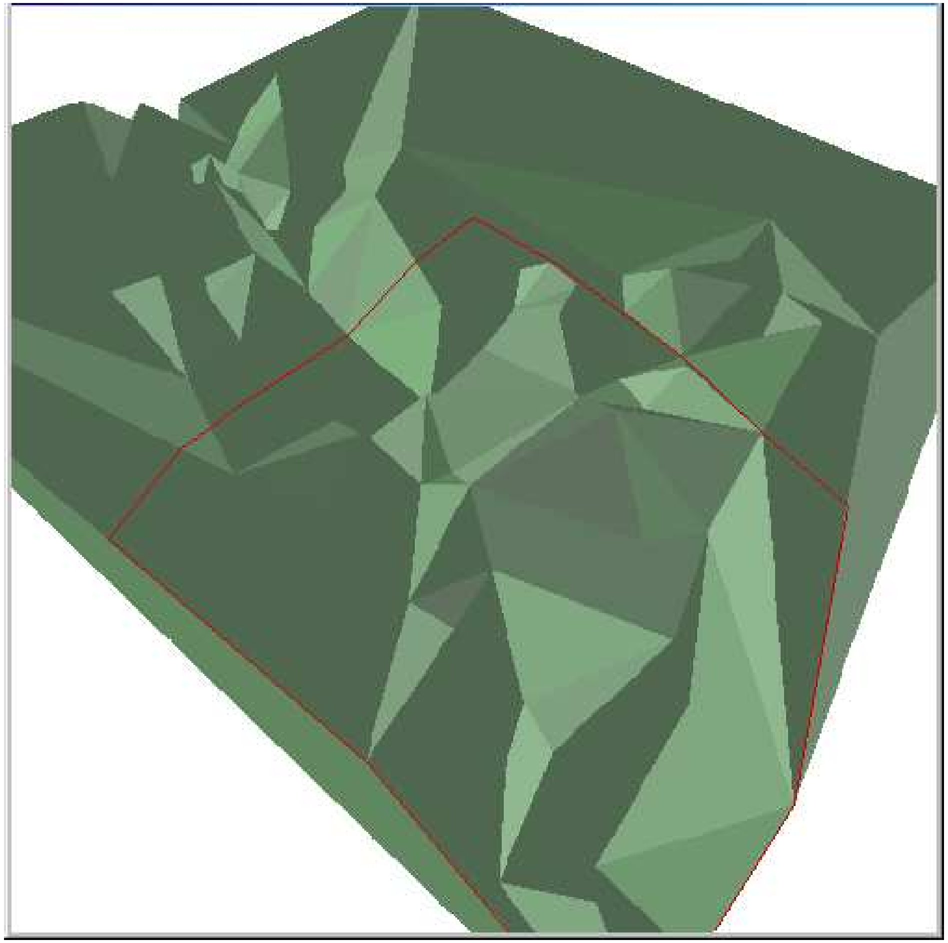
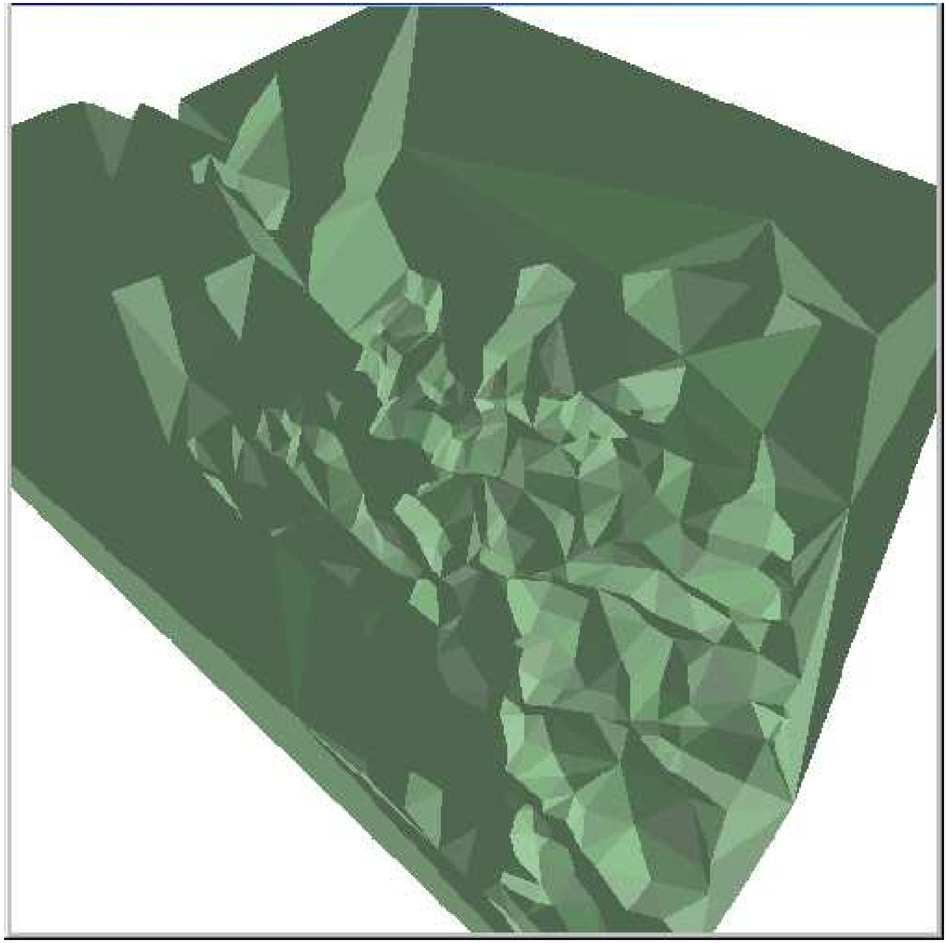
Burn-In

Coarse Approximation



Progressive Burn-In

Selected region for refinement



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- Preserve Geometry and Topology
- Optimal: performs at best bit rate?
- Image Encoder: Cohen,Dahmen, Daubechies,DeVore
- Burn In: DeVore, Johnson, Sharpley

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- Two essential ingredients

  a. metric ρ to measure distortion

  b. Precise definition of classes K_α to be compressed

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- smallest distortion for the given bit budget

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- Game: Find encoder/decoder E/D: for all values of n and all classes K_α, encoder is near optimal

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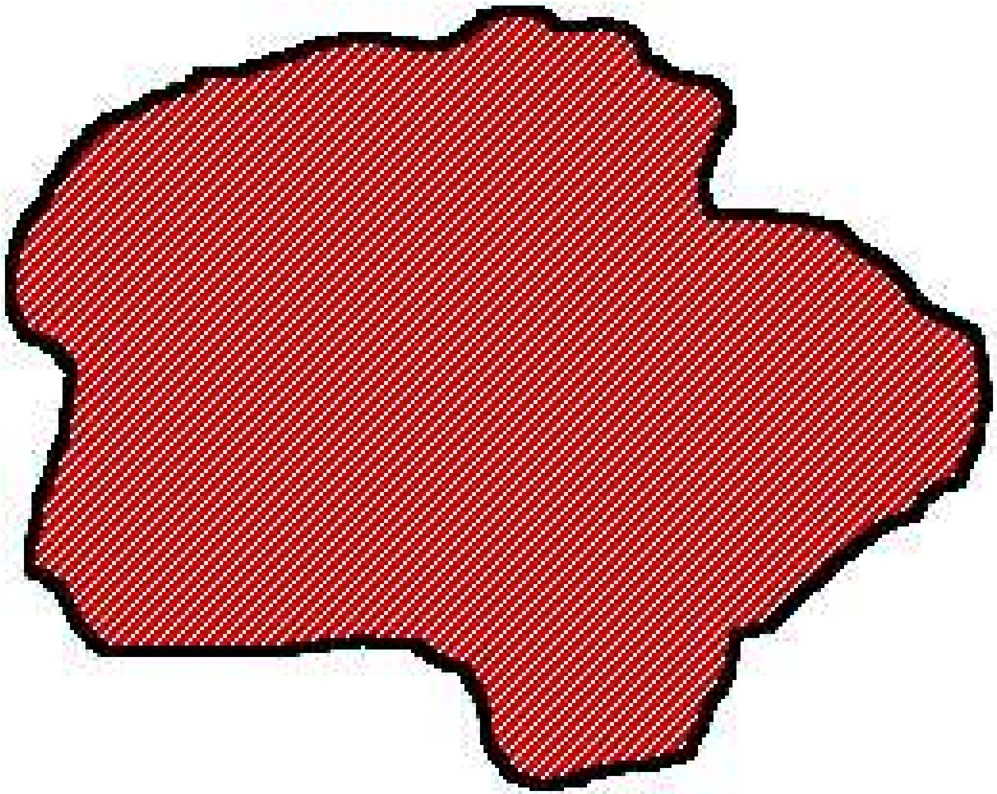
Covering



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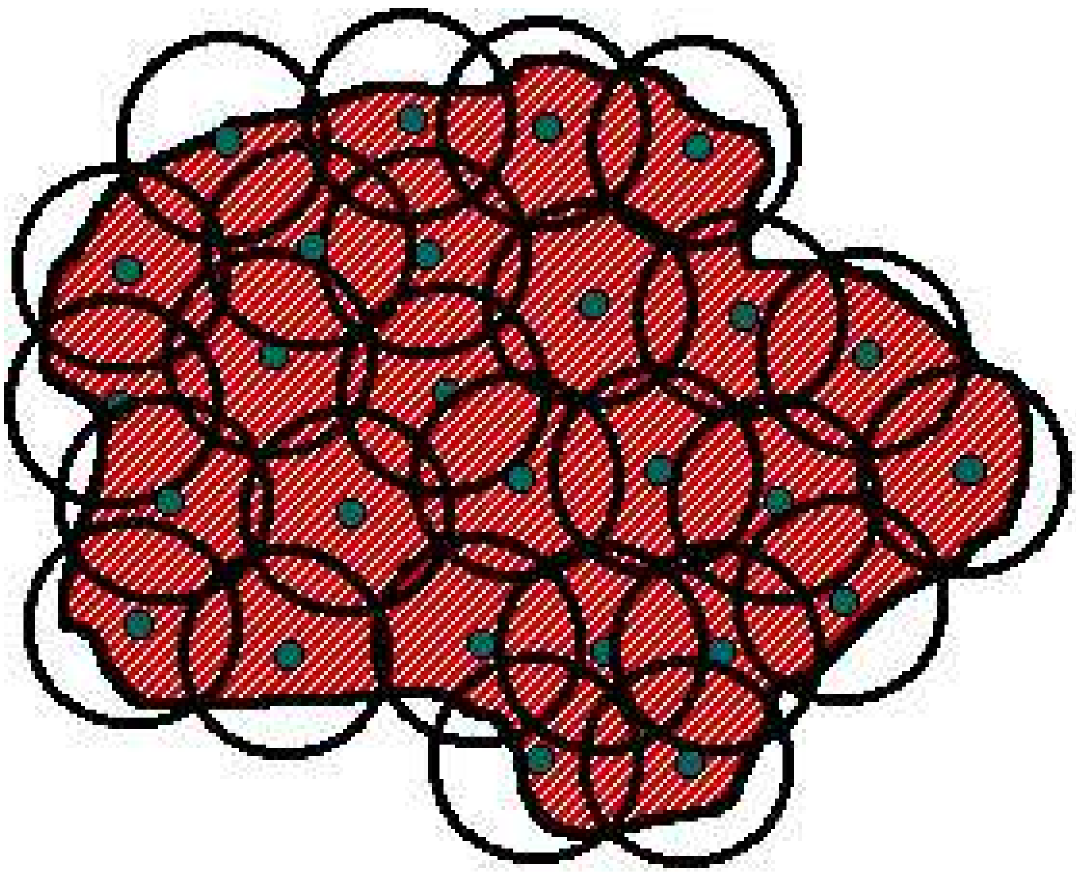

Kolmogorov Entropy

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Optimal Encoding: Kolmogorov Entropy

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- Minimal ϵ cover: $K \subset \cup_{i=1}^{N_\epsilon} \mathcal{B}(x_i, \epsilon)$
- $\delta_n(K) = \inf\{\epsilon : \; H_\epsilon(K) \leq n$
- Kolmogorov entropy of K gives our benchmark
- Usually not practical encoder

The Issues

1. The metric
2. The classes
3. Determine Entropy of Classes
4. Build near optimal Encoders/Decoders

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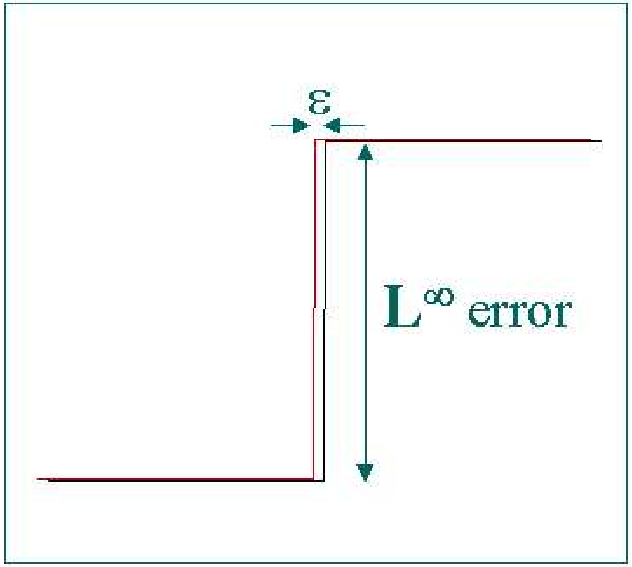

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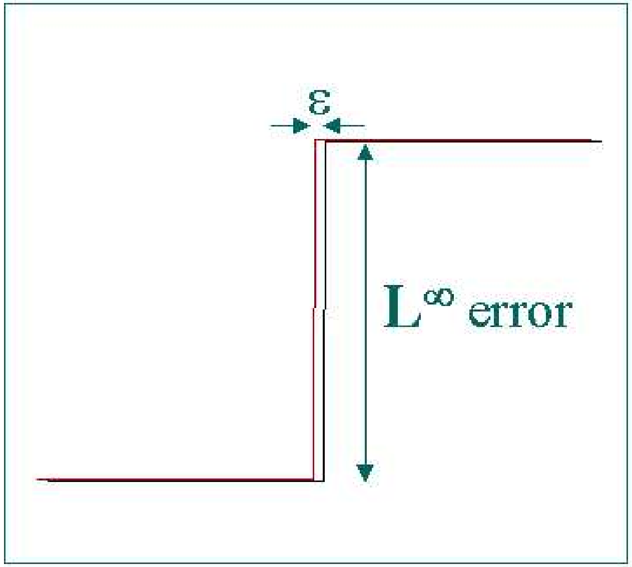
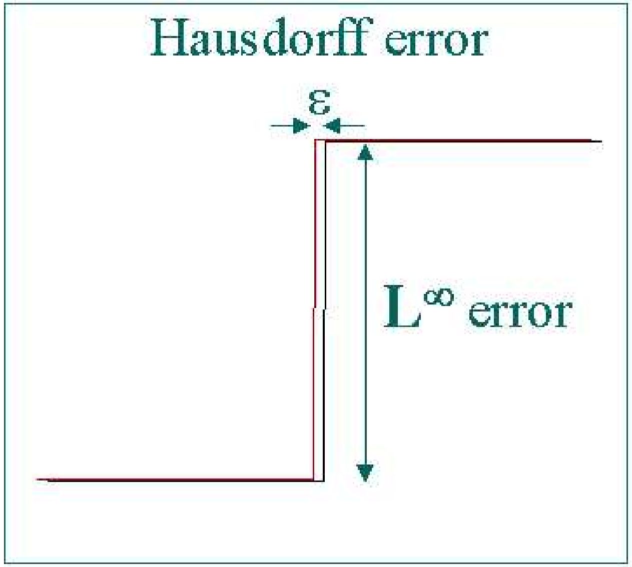


Hausdorff metric error



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- $\delta_H(S, S') := d(S, S') + d(S', S)$

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- metrics to incorporate line of sight

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- Differential Geometry to play crucial role

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- Not ready to formulate this

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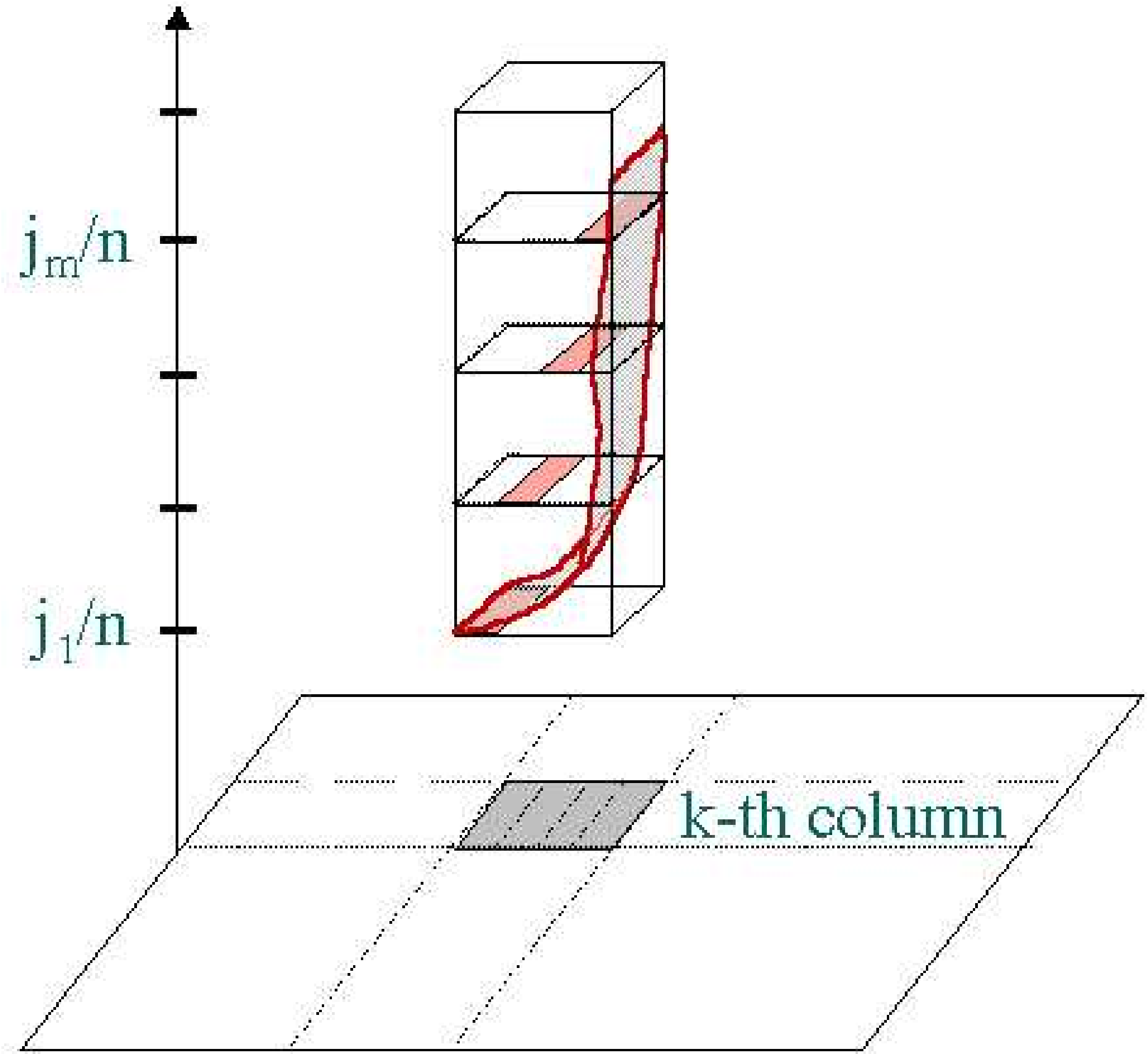
- $\delta_n(K) \leq Cn^{-1}$ for K ball in $\mathrm{BV}(\Omega)$, $\Omega \subset I\!R$
- $\delta_n(K) \leq Cn^{-2}$ for K class of continuous convex in $d = 1$.

box dimension



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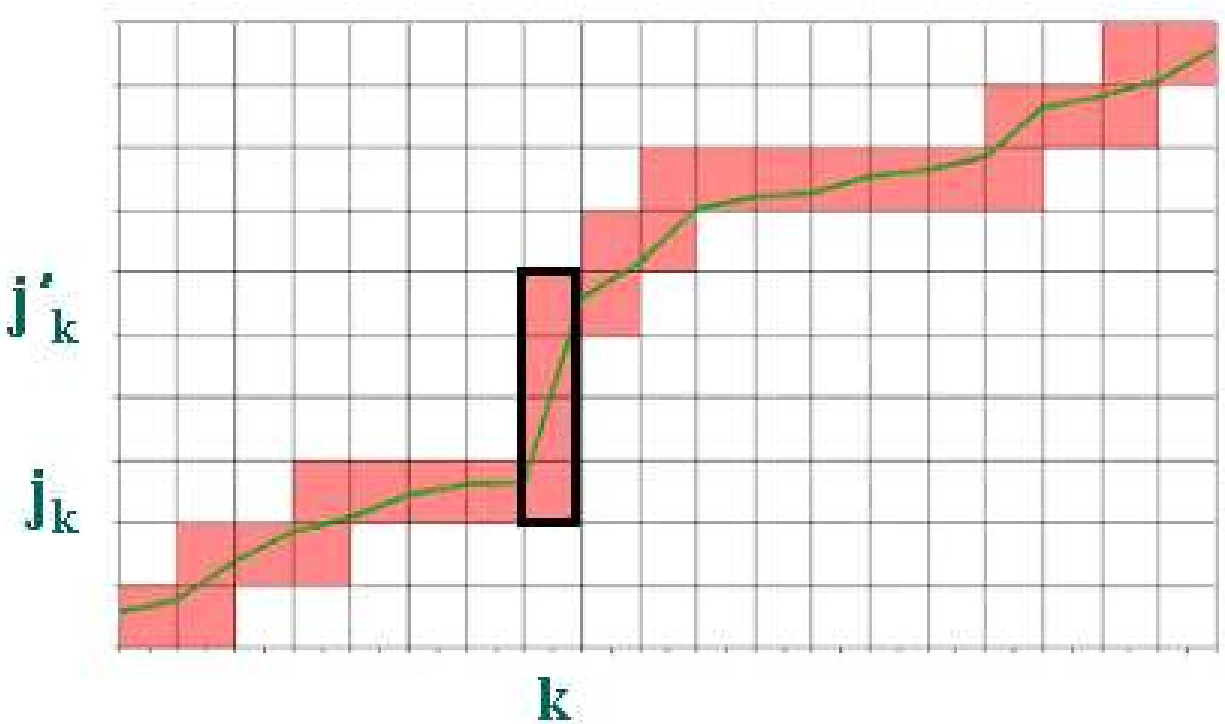

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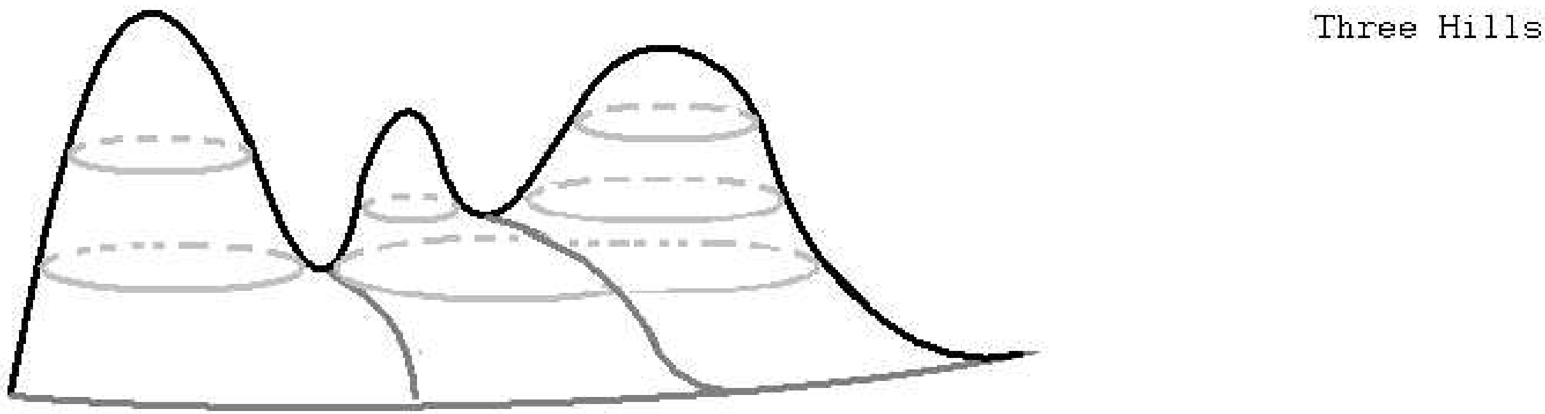
- continue

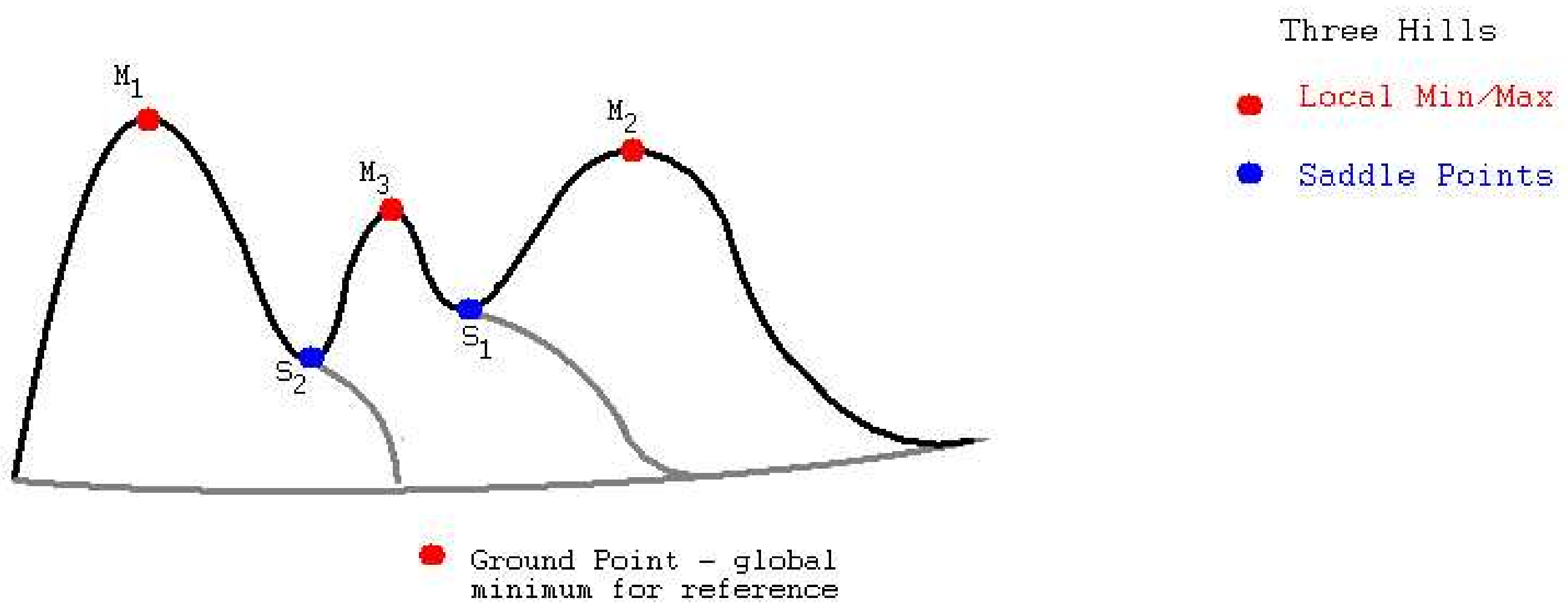
Morse structure and Reeb graphs

Original terrain



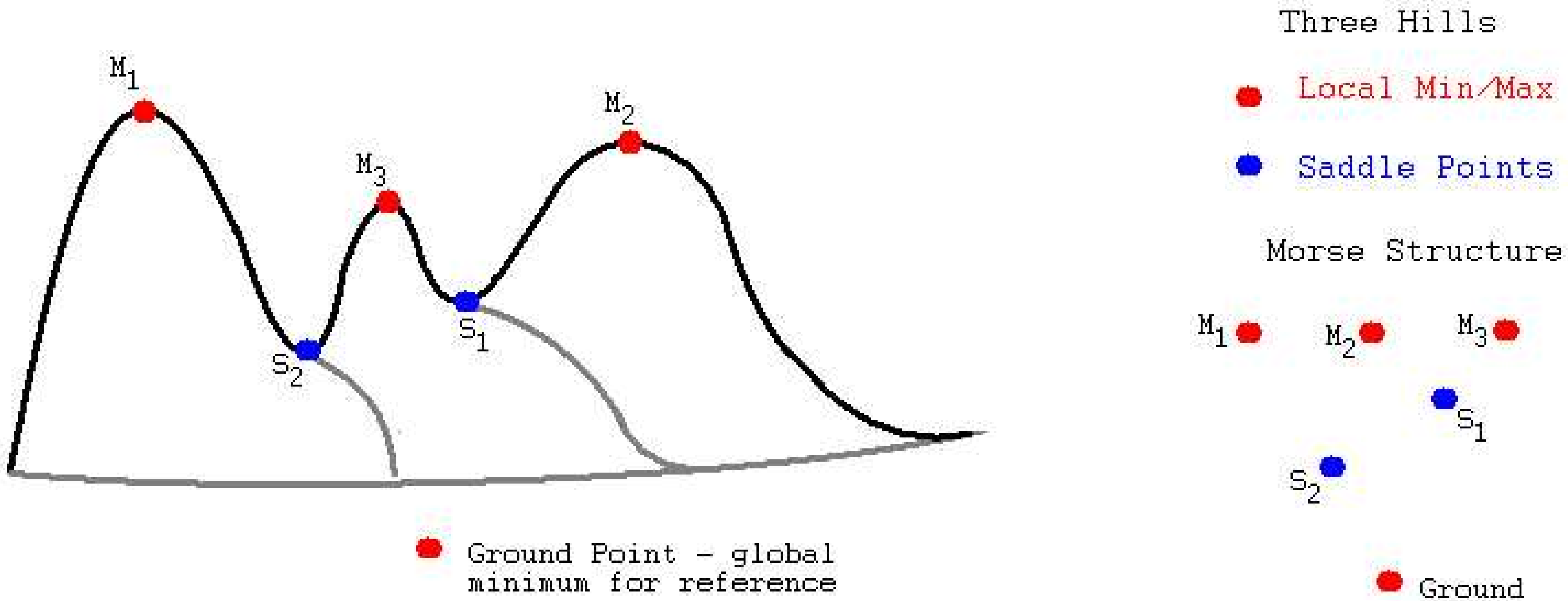
Morse structure and Reeb graphs

Select critical points



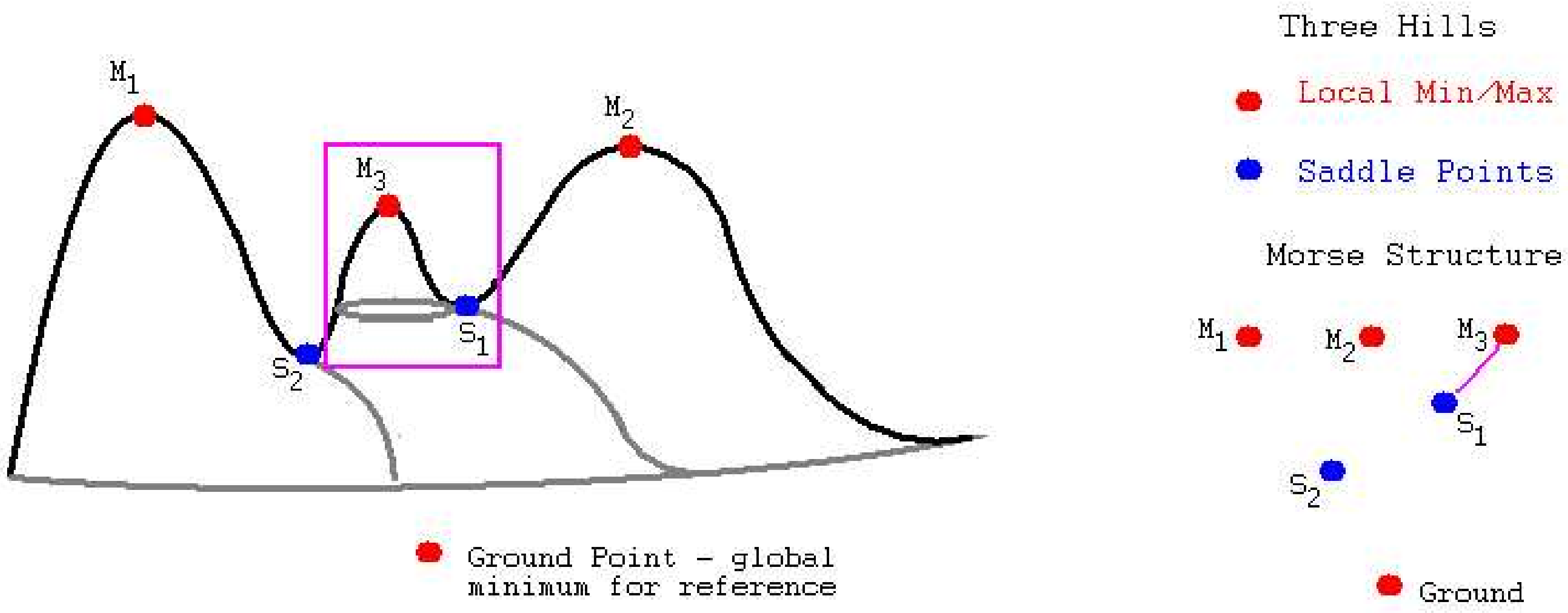
Morse structure and Reeb graphs

Represent as graph



Morse structure and Reeb graphs

Edge represents monotone region



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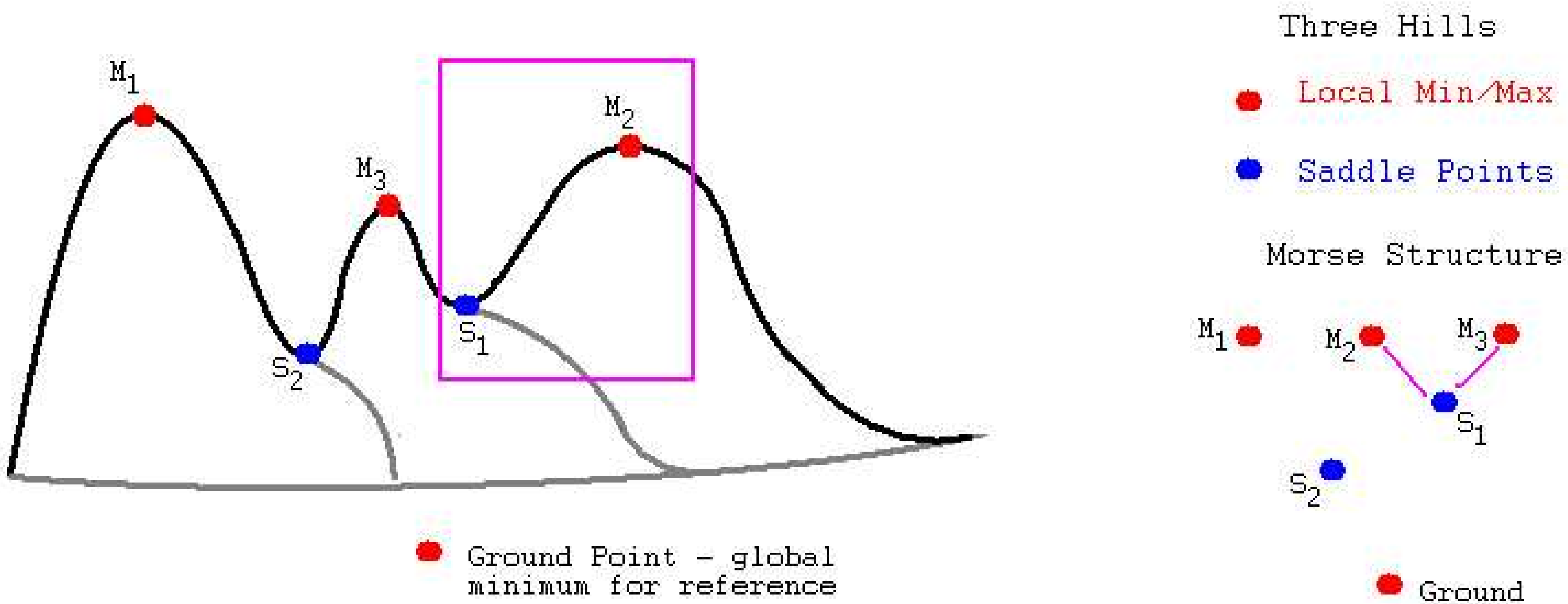

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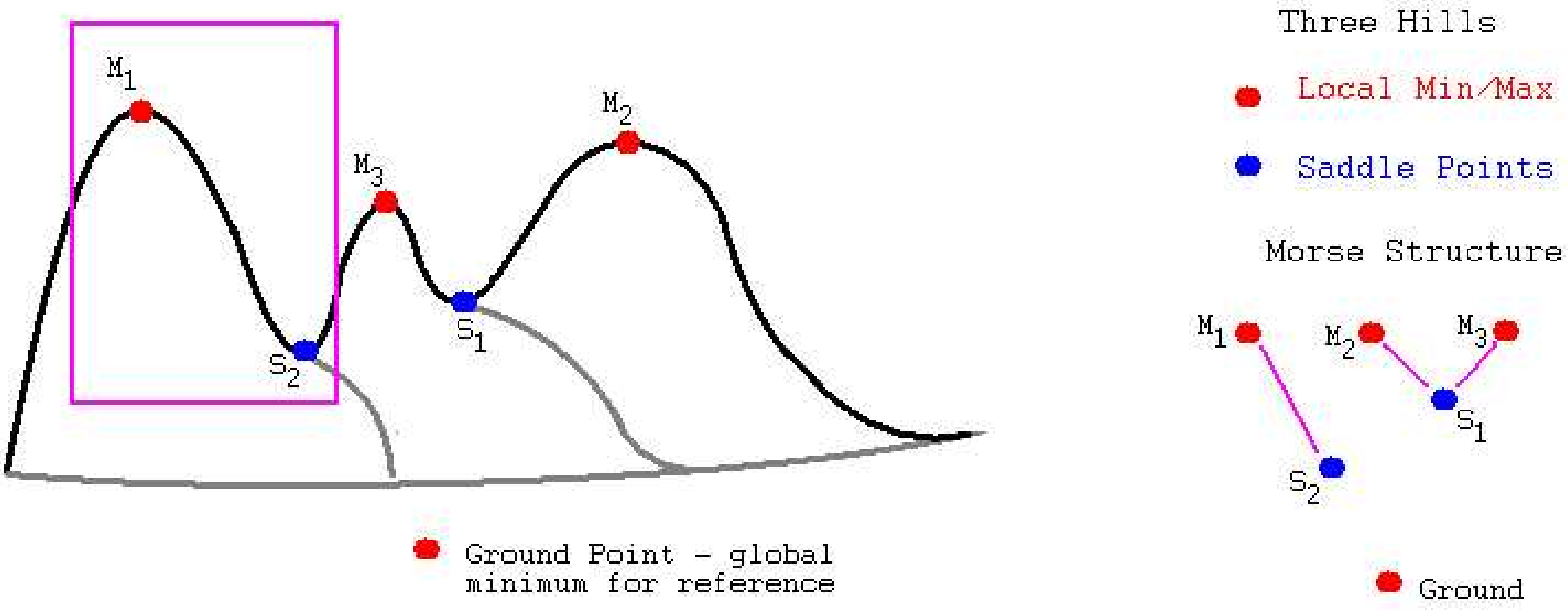
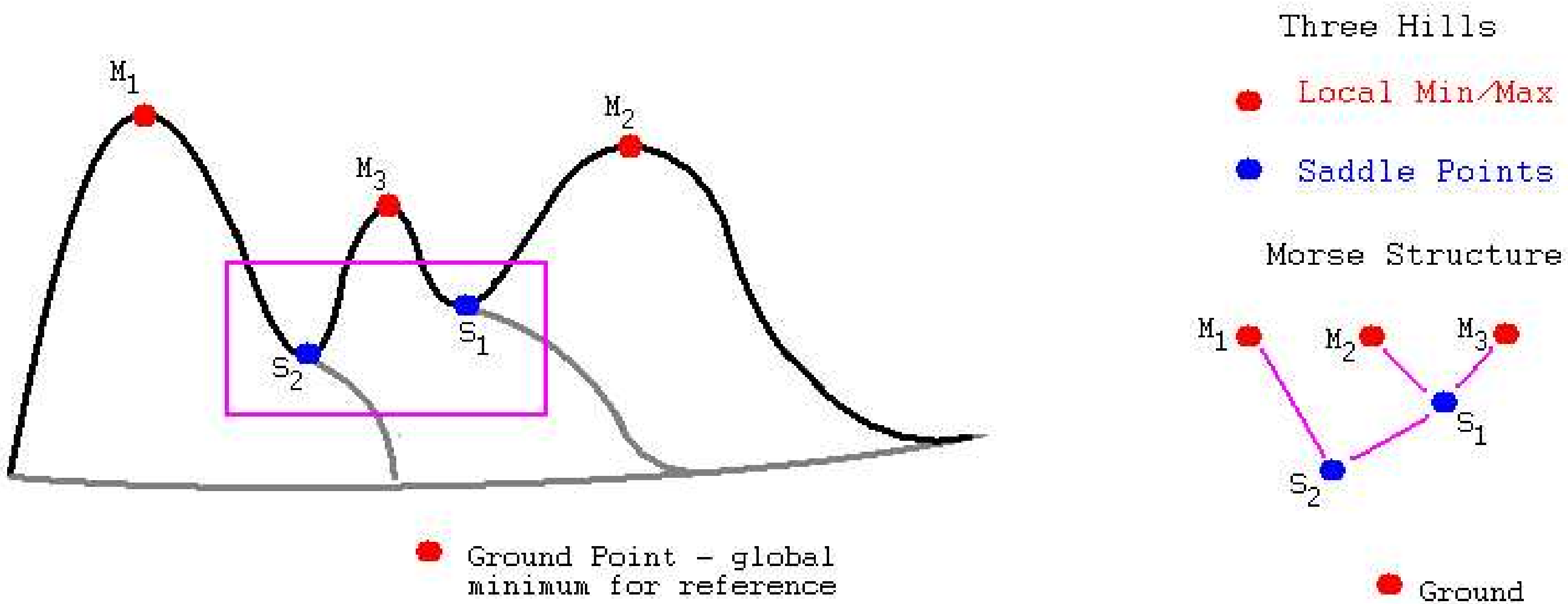


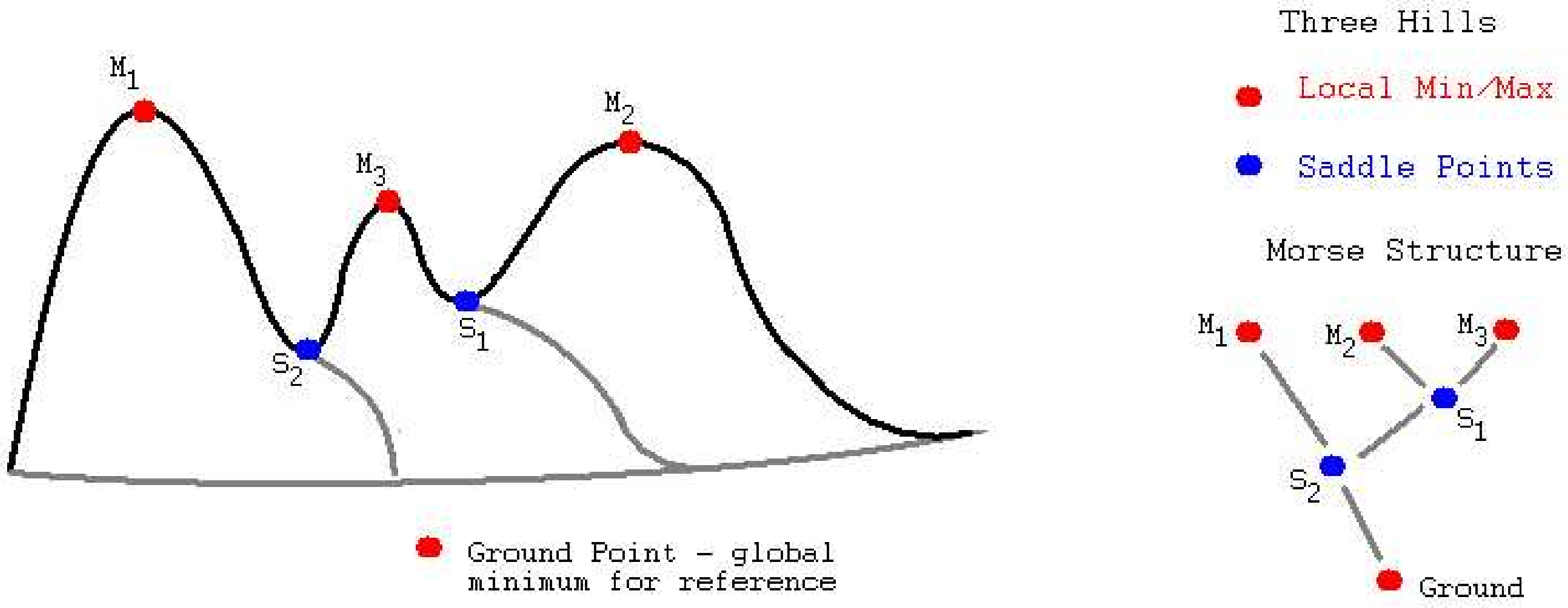
Morse structure and Reeb graphs

monotone region is a washer



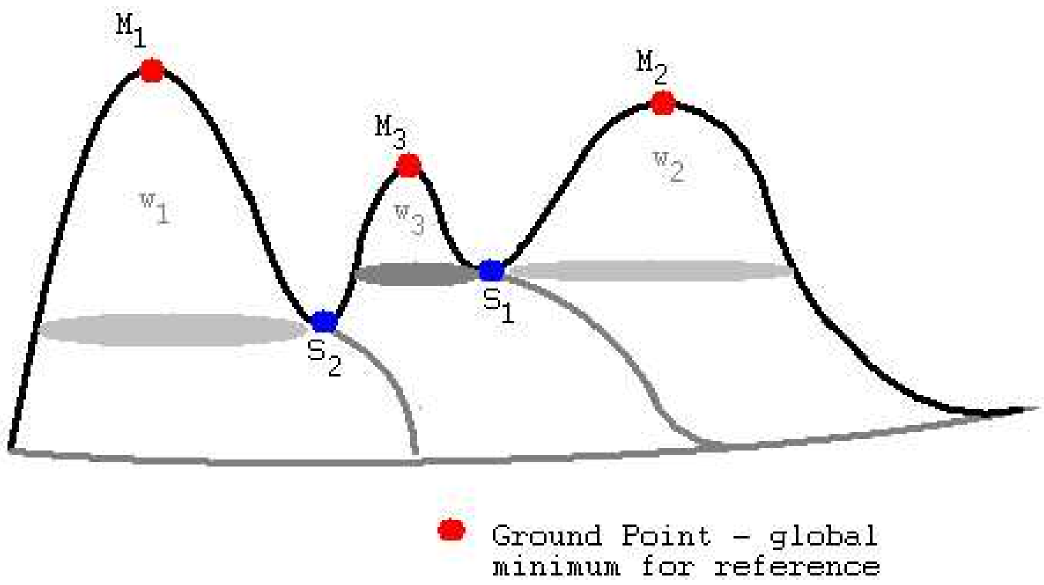
Morse structure and Reeb graphs

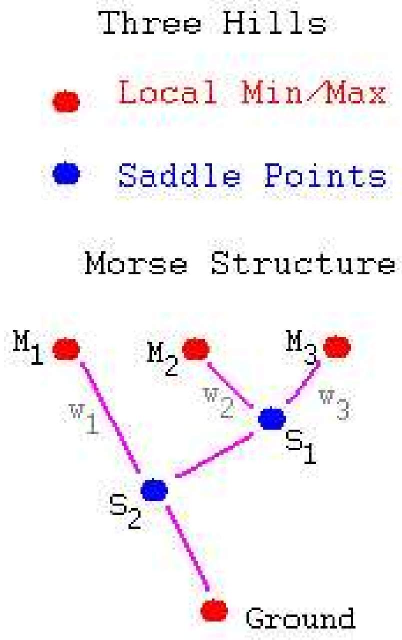
ground reference



Morse structure and Reeb graphs

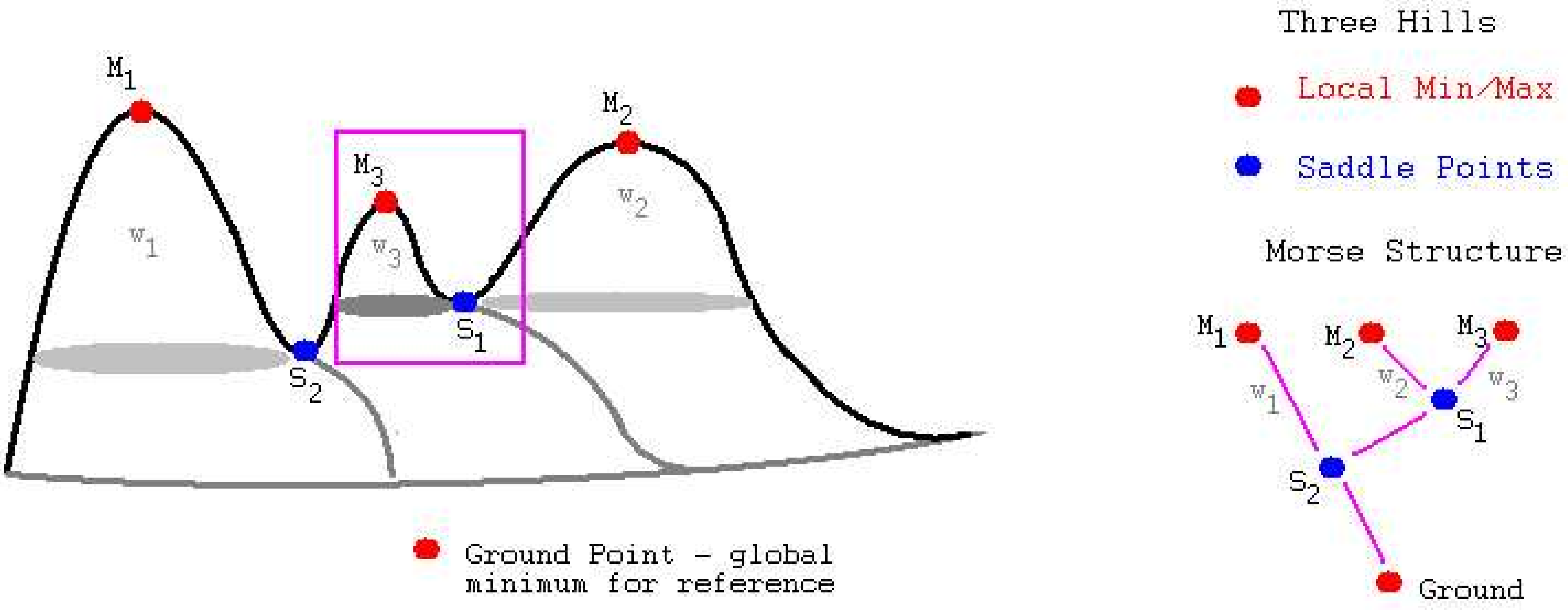
Each saddle point gives level curve





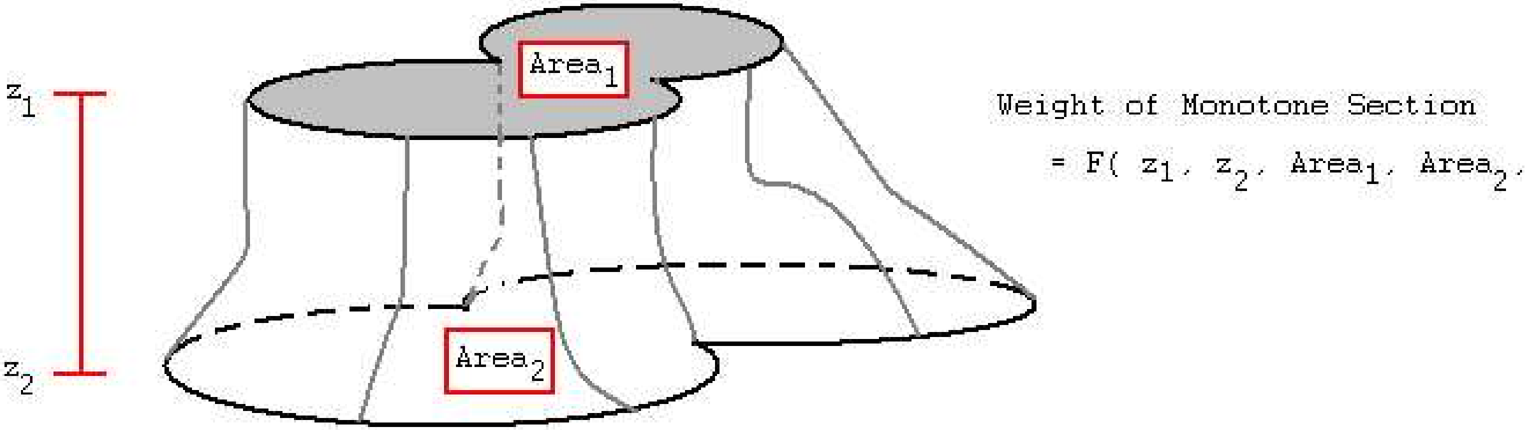
Morse structure and Reeb graphs

prioritize by assigning weights



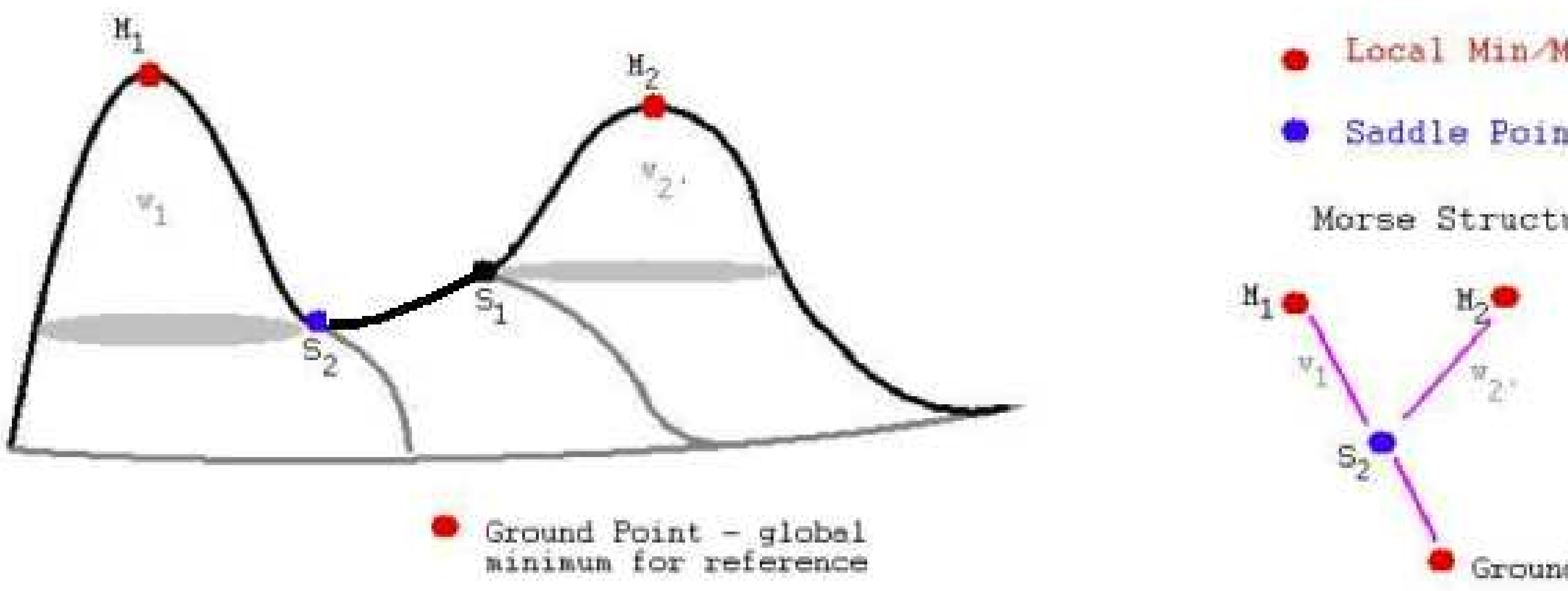
Morse structure and Reeb graphs

Definition of weight



Morse structure and Reeb graphs

removing low priority sections



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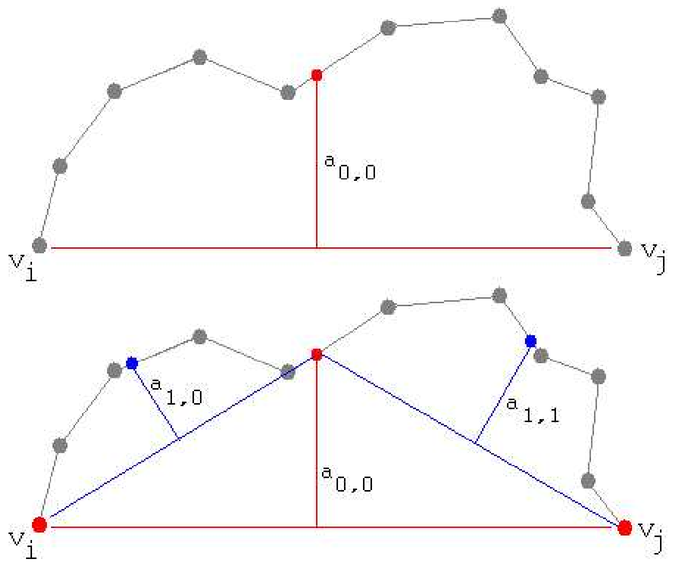
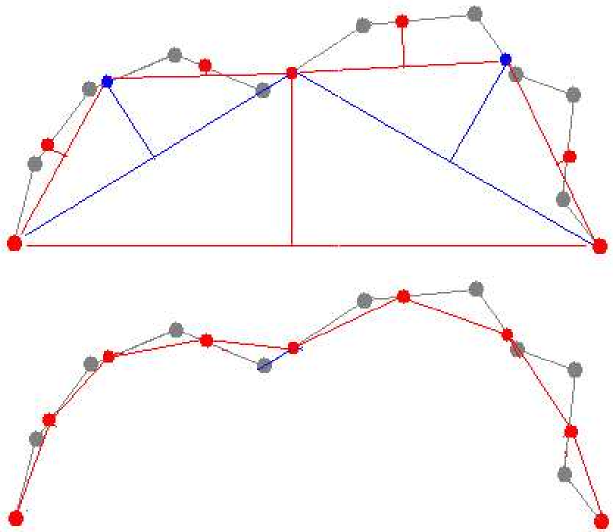
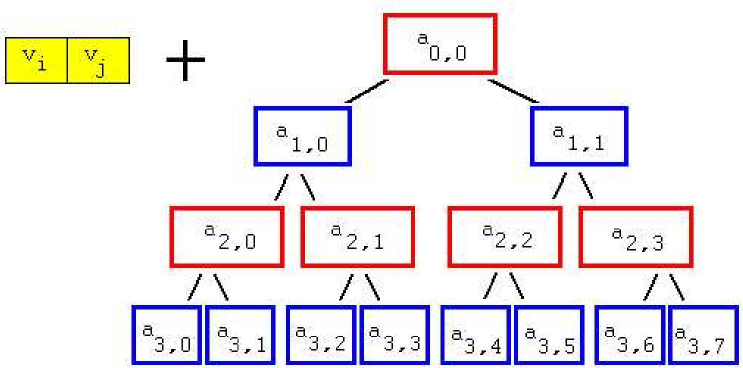
- K_2 includes all convex curves

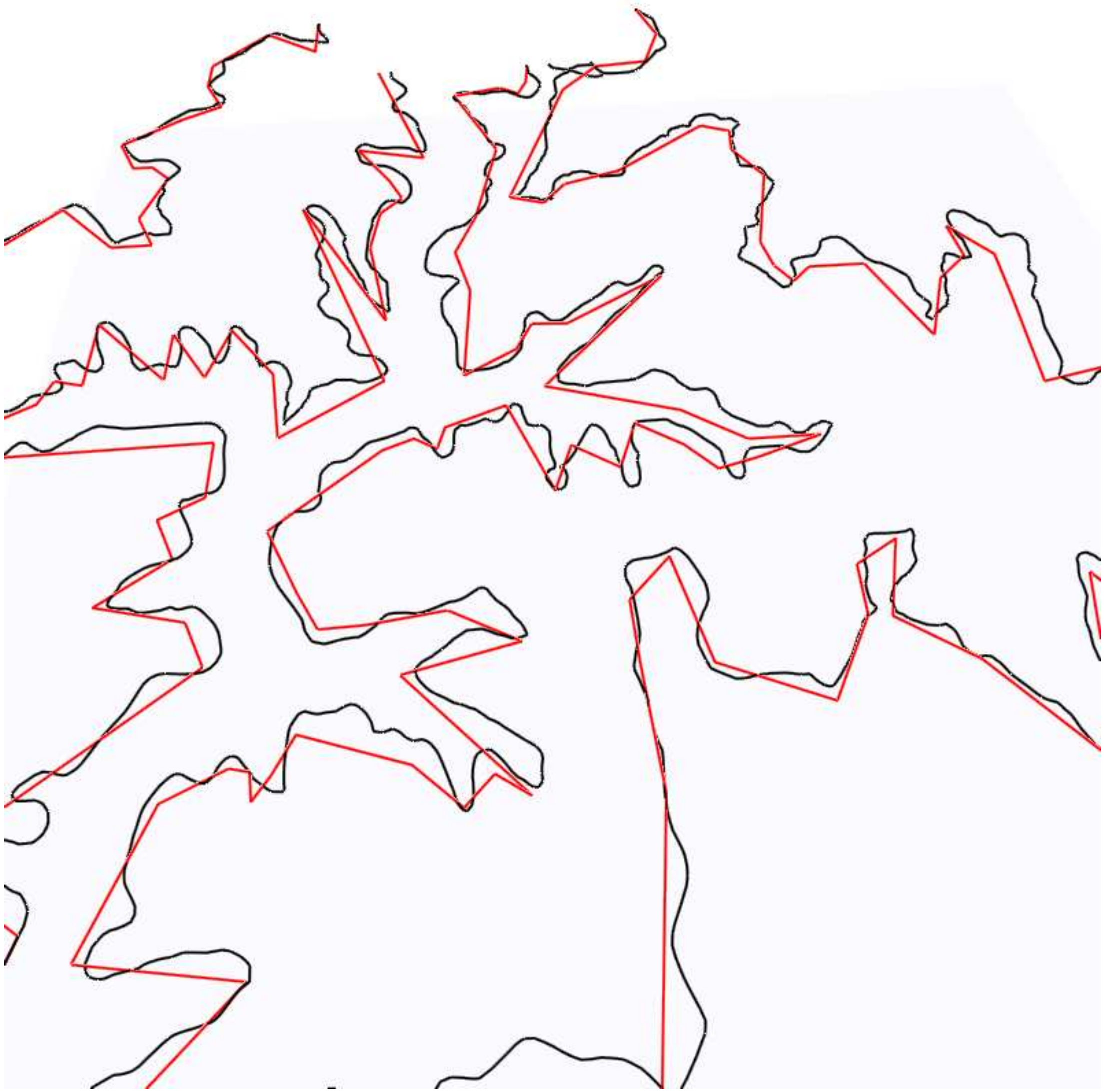
Simplest case - piecewise linear







Example: Level Curve Approximation



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- constrained Delaunay

References

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